INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

JUNIOR PAPER: YEARS 8,9,10

Tournament 40, Northern Autumn 2018 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. In triangle ABC, M is the midpoint of the side BC and E is a point on the side AC distinct from A and C. Suppose that $BE \geq 2AM$. Prove that one of the angles of triangle ABC is obtuse. (5 points)

2. There are 2018 people living on an island. Each person is one of: a knight, a knave, or a neither-knight-nor-knave. A knight always tells the truth, and a knave always lies. A neither-knight-nor-knave answers as the majority of people answered before him, or randomly, in the case that the numbers of *"Yes"* and *"No"* answers are equal. Everyone on the island knows which of the three possibilities each person is. One day all 2018 inhabitants of the island were arranged in a line and each in turn answered *"Yes"* or *"No"* to the same question:

   Are there more knights than knaves on the island?

   The total number of *"Yes"* answers was 1009 and everyone heard all the previous answers. Determine the maximum possible number of neither-knight-nor-knave people among the inhabitants of the island. (6 points)

3. One needs to write a number of the form $77\ldots7$, in base ten, using only 7s, the operations of addition, subtraction, multiplication, division, and raising to a power, and brackets. One can also use any number of 7s together with no operations between them. For the number 77 the shortest way to write it is to simply write 77. Does there exist a number of the form $77\ldots7$ that can be written under the rules above using a smaller number of 7s than in its base ten notation? (8 points)

4. A 7×7 grid board can be empty or can contain an invisible 2×2 ship that is located with its edges along the grid lines. A detector placed in a square of the board shows whether or not the square is occupied by the ship. All the detectors on the board are to be switched on at the same time. What is the smallest number of detectors needed to determine if the ship is on the board and, if so, exactly where it is located? (8 points)

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5. Let $ABCD$ be an isosceles trapezium (with AD parallel to BC), that is inscribed in a circle with centre O. The line BO and side AD meet at the point E. Suppose that O_1 and O_2 are the circumcentres of the triangles ABE and DBE, respectively. Prove that the points O_1, O_2, O and C are concyclic. (8 points)

6. Prove that

(a) any integer of the form $3k-2$, where k is an integer, can be represented as the sum of a perfect square and two perfect cubes of some integers. (7 points)

(b) any integer can be represented as the sum of a perfect square and three perfect cubes of some integers. (3 points)

7. There are $n \geq 2$ towns in some virtual world. Some pairs of towns are connected by roads, but there is no more than one road between any pair of towns. Any town can be reached from any other town via the roads. One can change a road only in a town. The world is called *simple*, if it is impossible to start at some town and to return to that same town without using the same road twice. Otherwise the world is called *complex*. Petya and Vasya play the following game:

> At the start Petya chooses a single direction on each road so that the road can be used in the chosen direction only and places a virtual tourist in one of the towns. Then Petya moves the tourist along a road in the permitted direction to a neighbouring town. On his turn, Vasya changes the permitted direction on one of the inbound or outbound roads of the town where the tourist is at the moment. Vasya wins if Petya cannot make a move.

Prove that

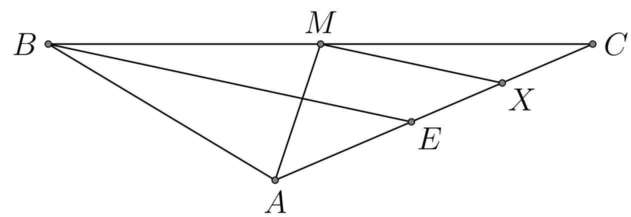
(a) in a simple world Petya can avoid defeat no matter how Vasya plays. (5 points)

(b) in a complex world Vasya can win for sure no matter how Petya plays. (7 points)

A Level Junior Paper Solutions

Prepared by Oleksiy Yevdokimov and Greg Gamble

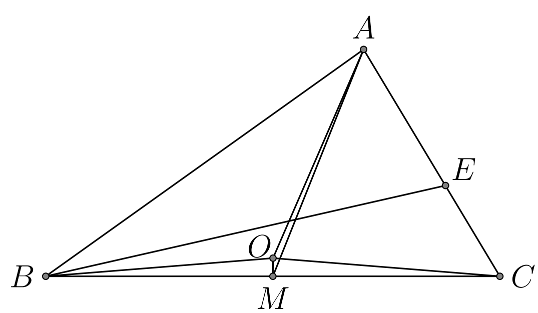
1. **Solution 1.** Let X be the midpoint of the line segment EC. Then, MX is a middle line of triangle BEC and $BE = 2MX$. Consider triangle AMC. Since the cevian MX must be shorter than at least one of the sides of triangle AMC that contain the vertex M, and $MX \geq MA$, we conclude that $MX < MC$. Hence, $MA < MC$ and so the point A is located inside the circle with diameter BC. So $\angle BAC$ is obtuse.



Solution 2. Suppose that the side BC of triangle ABC is the longest, but $\angle BAC$ is not obtuse. Then, the circumcentre O of triangle ABC either lies on BC and coincides with M (in the case $\angle BAC = 90°$) or O is located on the same side as A with respect to BC (in the case $\angle BAC < 90°$). So we have

$$2AM \geq 2AO = OB + OC \geq BC > BE,$$

contradicting $BE \leq 2AM$. Hence, $\angle BAC$ is obtuse.



2. The maximum possible number of neither-knight-nor-knaves among the inhabitants of the island is 1009. First, we make an estimation and show that the maximum number of neither-knight-nor-knave is bounded above by 1009 and then give an example to show that 1009 neither-knight-nor-knaves is possible. Note that there were 1009 *"No"* answers in all, since there were 1009 *"Yes"* answers in all. Let m be the smaller of the number of *"Yes"* answers and the number of *"No"* answers at a given stage. When giving his answer each neither-knight-nor-knave cannot increase m. Thus at each stage at least m of those who have given their answer are not neither-knight-nor-knaves. Since after all the inhabitants have answered $m = 1009$, at least 1009 of the inhabitants are not neither-knight-nor-knaves. Hence the number of neither-knight-nor-knaves is at most $2018 - 1009 = 1009$.

So we are left now with showing that the bound of 1009 neither-knight-nor-knaves is possible. Indeed, the first 1009 inhabitants in the line could be neither-knight-nor-knaves who also answer *"No"*, followed by 1009 knights who all answer *"Yes"*.

Thus, the maximum possible number of neither-knight-nor-knaves is 1009.

Note. Other examples with 1009 neither-knight-nor-knaves exist. There is the complementary possibility where the first 1009 inhabitants in line are neither-knight-nor-knaves who all answer *"Yes"*, followed by 1009 knaves who all answer *"No"*. Another possibility is that 1008 knights (who say *"Yes"*) are interspersed among 1008 neither-knight-nor-knaves in such a way that they still say *"No"*, e.g. every second person in line is a knight, followed by a neither-knight-nor-knave who (randomly) says *"Yes"*, followed by a knave who says *"No"*.

3. **Solution 1.** Yes, it is possible, via the following observations.

$$\underbrace{777\ldots7}_{n\text{ digits}} = 7\cdot\underbrace{111\ldots1}_{n\text{ digits}} = \frac{7\cdot(10^n-1)}{9} = \frac{7\cdot 10^n - 7}{9} = \frac{7\cdot\left(\frac{77-7}{7}\right)^n - 7}{7+\frac{7+7}{7}},$$

where, for example, we can use $n = 77$ or $n = 14 = 7+7$. For $n = 77$, the first expression has 77 7s, whereas the final expression has only twelve 7s.

Note. A 2-digit number $(77-7)/7 = 10$ was used in the representation above. Replacing $(77-7)/7$ with $7+(7+7+7)/7$ we can get an example with more 7s, but without any 2-digit number; just with a 1-digit number 7 involved.

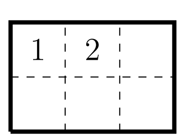
Solution 2 by Budun Budunov. Since $\underbrace{77\ldots7}_{2n\text{ digits}} = \underbrace{77\ldots7}_{n\text{ digits}}\cdot(10^n+1)$, we obtain

$$\underbrace{77\ldots7}_{28\text{ digits}} = \underbrace{77\ldots7}_{14\text{ digits}}\cdot\left(\left(\frac{77-7}{7}\right)^{7+7} + \frac{7}{7}\right).$$

Note. Any number of the form $77\ldots7$ with more than one 7 (e.g. 77 and so on) can be used as the exponent in both this solution and Solution 1.

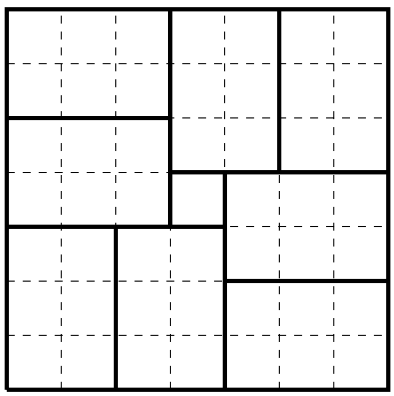
4. **Solution 1 by William Steinberg.** We will show that the smallest number of detectors needed to both determine if a ship is on the board and, if so, determine its exact location, is 16. Our strategy is to first determine a lower bound, and then provide an example to show the lower bound is achievable and hence is the required minimum.

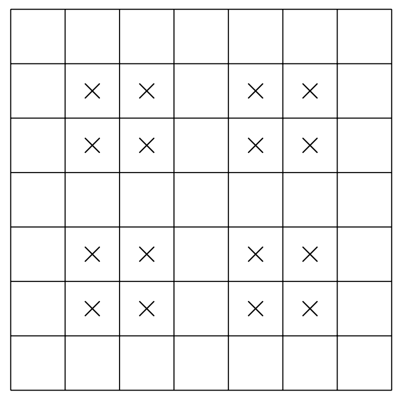
First we show that in every 2×3 subgrid we need at least 2 detectors.



Without loss of generality, orientate the subgrid so that it has 2 rows and 3 columns as shown. First suppose one detector is enough. If the detector is in a corner cell, without loss of generality the cell numbered 1, then a ship can take up the remaining 2 columns undetected. If the detector is in a middle column cell, without loss of generality in cell 2, then a ship can be detected but could take up the first two columns or the last 2 columns, i.e. its location cannot be exactly determined. Thus in every 2×3 subgrid we need at least 2 detectors.

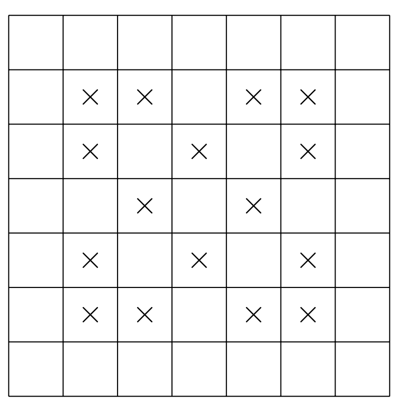
As shown in the diagram below left the board can be divided into eight 2×3 rectangles and one square in the centre. For a ship to be detected and located with certainty, we require at least 2 detectors in each 2×3 rectangle. Thus, at least 16 detectors are needed.





The diagram above right shows an example with 16 detectors (indicated by $\times$s) in the cells shown. If a ship is present it must intersect one of the 2×2 blocks of crossed squares. Indeed, a ship must intersect exactly one crossed square, exactly two crossed squares, or exactly four crossed squares. In each case the precise location of the ship or its absence can be determined.

Alternative detector configuration. The diagram below shows another example with 16 detectors.



Again, if a ship is present it must intersect at least one of the crossed squares. Indeed, a ship must intersect exactly one crossed square, exactly two crossed squares, or exactly three crossed squares. In each case the precise location of the ship or its absence can be determined.

Note 1. There are no other ways where 16 detectors can be located on the board.

Note 2. Even if it is known that the ship is definitely on the board, i.e. the board is not empty, 16 detectors are still needed to determine the location of the ship.

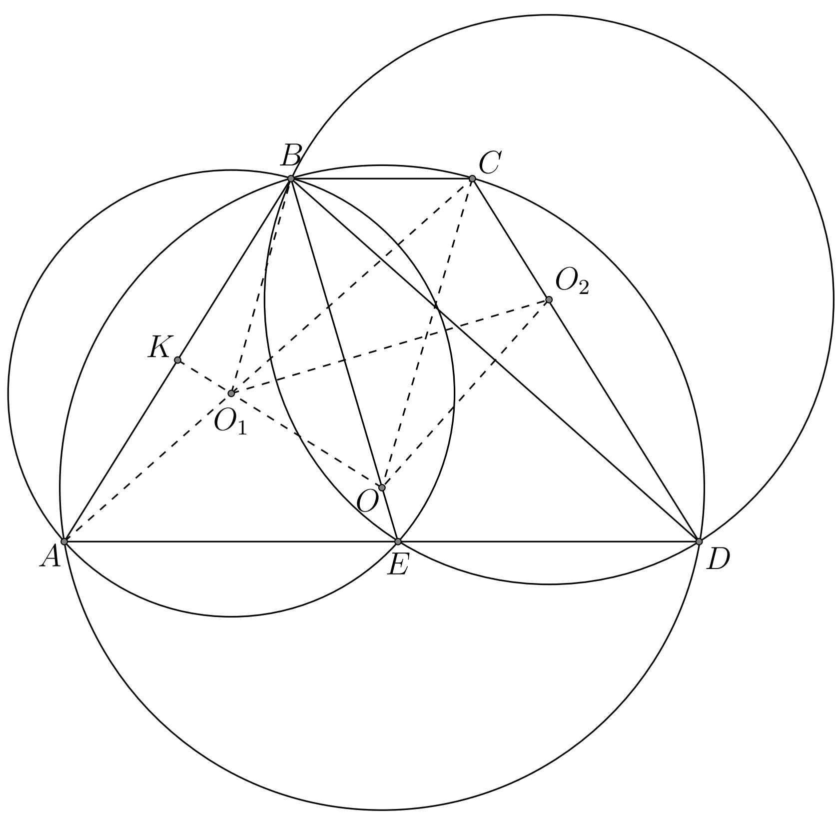
5. **Solution 1.** Let K be the midpoint of AB. Let Γ, Γ_1 and Γ_2 be the circumcircles of $ABCD$, ABE and DBE, respectively, and note they have centres O, O_1 and O_2, respectively. Since Γ and Γ_1 intersect at points A and B, the line OO_1 is the perpendicular bisector of AB. Similarly, OO_2 is the perpendicular bisector of BD, and O_1O_2 is the perpendicular bisector of BE. Exploiting the axis of symmetry property of perpendicular bisectors, we have

$$\begin{aligned}\angle BO_1O_2 &= \tfrac{1}{2}\angle BO_1E, \text{ since } O_1O_2 \text{ is an axis of symmetry}\\ &= \angle BAE, \quad \text{since half central angle equals inscribed angle, in } \Gamma_1\\ &= \angle BAD, \quad \text{same angle}\\ &= \tfrac{1}{2}\angle BOD, \quad \text{since inscribed angle equals half central angle, in } \Gamma\\ &= \angle BOO_2, \quad \text{since } OO_2 \text{ is an axis of symmetry.}\end{aligned}$$

Thus, quadrilateral BO_1OO_2 is cyclic.

$$\begin{aligned}\angle KO_1B &= \tfrac{1}{2}\angle AO_1B, \text{ since } KO_1 \text{ is perpendicular bisector of } AB\\ &= \angle AEB, \quad \text{since half central angle equals inscribed angle in } \Gamma\\ &= \angle CBE, \quad \text{alternate angles, since } AE \parallel BC\\ &= \angle CBO, \quad \text{same angle}\\ &= \angle BCO, \quad \text{since } BOC \text{ is isosceles, legs } OB,\, OC \text{ being radii of } \Gamma.\end{aligned}$$

Thus, quadrilateral BO_1OC is cyclic, since $\angle KO_1B$ is the exterior angle opposite $\angle BCO$. But, the circumcircles of BO_1OO_2 and BO_1OC are the same circle, since points B, O_1, O are common and three noncollinear points are sufficient to determine a circle. Hence, points O_1, O_2, O and C are concyclic.



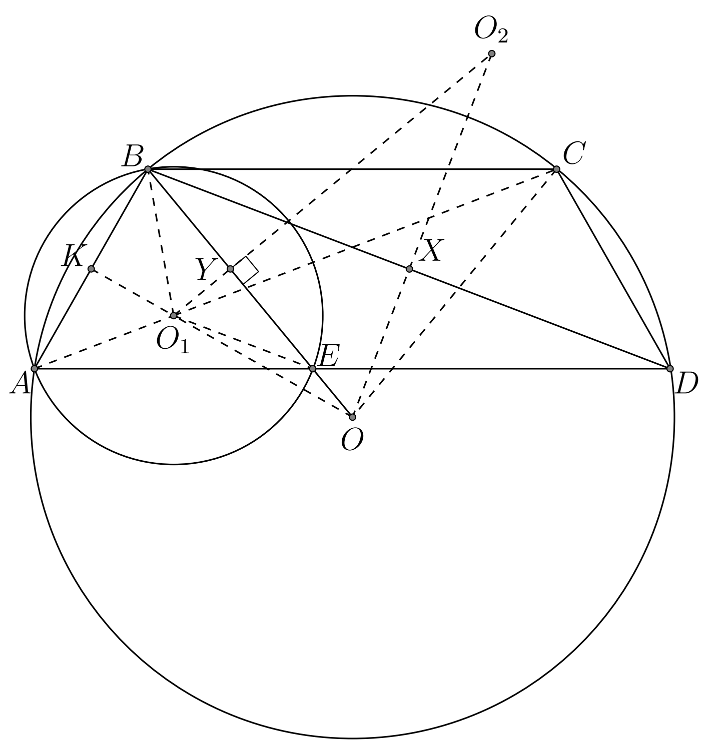
Solution 2. As in Solution 1., let K be the midpoint of AB; and let Γ, Γ_1 and Γ_2 be the circumcircles of $ABCD$, ABE and DBE, respectively, whose centres are O, O_1 and O_2, respectively; and deduce that OO_1, OO_2 and O_1O_2 are the perpendicular bisectors of AB, BD and BE, respectively. Also, let X be the point of intersection of BD and OO_2, and let Y be the point of intersection of BE and O_1O_2. We claim O_1 lies on AC. Indeed,

$$\begin{aligned}
\angle O_1AE &= \tfrac{1}{2}(180^\circ - \angle AO_1E), \text{ since triangle } AO_1E \text{ is isosceles} \\
&= 90^\circ - \tfrac{1}{2}\angle AO_1E \\
&= 90^\circ - \angle ABE, \text{ since half central angle equals inscribed angle, in } \Gamma_1 \\
&= 90^\circ - \angle KBO, \quad \text{same angle} \\
&= \angle KOB, \text{ since } \angle BKO = 90^\circ \text{ in triangle } KOB \\
&= \tfrac{1}{2}\angle AOB, \text{ since } OO_1 \text{ is an axis of symmetry} \\
&= \angle ADB, \text{ since half central angle equals inscribed angle, in } \Gamma \\
&= \angle CAD, \text{ since } ABCD \text{ is an isosceles trapezium.}
\end{aligned}$$

Hence, O_1 lies on AC. Therefore,

$$\begin{aligned}
\angle OCO_1 &= \angle OCA, \quad \text{same angle} \\
&= \angle OBD, \quad \text{since } ABCD \text{ is an isosceles trapezium} \\
&= \angle OBX, \quad \text{same angle} \\
&= 90^\circ - \angle BOX, \text{ since } \angle BXO = 90^\circ \text{ in triangle } BOX \\
&= 90^\circ - \angle YOO_2, \text{ same angle} \\
&= \angle YO_2O, \text{ since } \angle O_2YO = 90^\circ \text{ in triangle } YO_2O_1 \\
&= \angle OO_2O_1.
\end{aligned}$$

and hence, OCO_2O_1 is cyclic, i.e. points O_1, O_2, O and C are concyclic.



Note. The fact that O_1 lies on AC, as proved in Solution 2., can be proven in another way. Let diagonal AC intersect the circumcircle of triangle OAE at point P. We show that P coincides with O_1. Firstly,

$$\begin{aligned}
\angle PEA &= \angle BEA - \angle OEP \\
&= \angle EBC - \angle OEP, \text{ since } \angle BEA = \angle EBC \text{ are alternate angles} \\
&= \angle OCB - \angle OEP, \text{ since } \angle EBC = \angle OBC = \angle OCB \\
&= \angle OCB - \angle OCP, \text{ since } \angle OEP = \angle OAP = \angle OCP \\
&= \angle ACB \\
&= \angle CAE, \quad \text{alternate angles} \\
&= \angle PAE, \quad \text{same angle.}
\end{aligned}$$

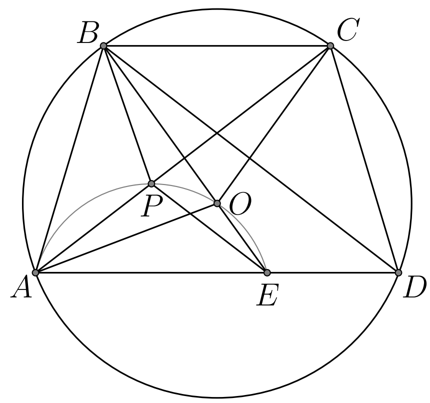
Hence, triangle APE is isosceles, and so, $PA = PE$.

Since $\angle POB$ is the exterior angle opposite $\angle PAE$ in cyclic quadrilateral $PAOE$,

$$\angle BOP = \angle PAE = \angle PEA = \angle POA.$$

Thus, with common side PO, and $OB = OA$ (radii of Γ), triangles POB and POA are congruent (by the SAS Rule).

Hence, $PB = PA = PE$, which means P is the circumcentre of triangle ABE and so $P = O_1$.



6. (a) The required representation follows from the following identity,

$$3k - 2 = k^3 - (k+3)^3 + (3k+5)^2 = k^3 + (-(k+3))^3 + (3k+5)^2.$$

   (b) First, we bring any given integer to the form of $3k - 2$ by subtracting an appropriate perfect cube, which can be 0, 1 or -1, and then apply the identity in (a).

7. Consider a graph, where vertices are towns and edges are roads.

   (a) For a *simple* world, the graph is a tree, i.e. a connected simple graph without cycles (a cycle is a sequence of distinct adjacent vertices that begins and ends at the same vertex). Petya can choose any vertex. From any other vertex there exists exactly one path to the chosen vertex. On each path to the chosen town, Petya chooses all directions towards the chosen town. At the start, Petya moves the tourist to the chosen town (from a neighbouring one). All roads to the chosen town are inbound. On his turn, Vasya changes the permitted direction of one of the roads. Then Petya moves the tourist along the road for which Vasya has changed the direction. All the roads to the town the tourist has just arrived at are inbound. Vasya can change the permitted direction of one of the roads again and Petya moves the tourist to a neighbouring town along that road. All roads to the new town the tourist has just arrived at are inbound again and the situation is repeated again and again. So Petya can always make a move and avoid defeat in a simple world no matter how Vasya plays.

(b) For a *complex* world, the graph is a connected graph with cycles. We use induction on the number of vertices (towns). The base case is a simple cycle. Denote vertices (towns) by $A_1, A_2, \ldots$. Assume that in a simple cycle $A_1 A_2 \ldots A_n$ all directions are arranged in some way and the tourist arrives at A_2 from A_1. Then Vasya will change the direction of the edge the tourist hasn't used yet (i.e. in front of the tourist). In other words, Vasya's strategy should be to disallow the tourist from moving back. Assume the tourist has been able to get to A_1. Then Vasya changes the direction from outbound to inbound in front of the tourist so that no moves are available. Vasya wins.

Now, we consider the inductive step. The graph is not a simple cycle. Choose a cycle of the minimal length in the graph. Call the cycle with the minimal length C. C is a simple graph and doesn't contain any edge inside itself. So there are vertices outside of C. Choose a vertex V of maximum distance from C. Denote a graph without V by G. G is a connected graph and G contains a cycle. According to the inductive assumption, Vasya has winning strategy in G for any direction of edges in G. Thus, inside G Vasya follows his winning strategy. Since Petya loses in G, sooner or later the tourist will be forced to move to V. Then, Vasya will change the inbound edge (road) the tourist moved into V to be the outbound one. The tourist will depart from V and Vasya should make any acceptable move in G next. So the tourist is in G again, where Vasya has winning strategy. That means the tourist sooner or later will be forced to move to V again and the number of inbound edges to V be decreased every time when the tourist comes back to V. Since the number of edges (roads) to V is finite, the tourist finally will be unable to get to V and lose inevitably inside G anyway.

Thus, Vasya can win for sure in a complex world no matter how Petya plays.